Exercise 85

Find a parabola $y = ax^2 + bx + c$ that passes through the point $(1, 4)$ and whose tangent lines at $x = -1$ and $x = 5$ have slopes 6 and -2, respectively.

Solution

The fact that the parabola passes through $(1, 4)$ means that

$$4 = a(1)^2 + b(1) + c = a + b + c. \tag{1}$$

Take the derivative of the equation for the parabola.

$$\begin{aligned} y' &= \frac{d}{dx}(ax^2 + bx + c) \\ &= 2ax + b \end{aligned}$$

Since the tangent lines at $x = -1$ and $x = 5$ have slopes 6 and -2, respectively,

$$\begin{cases} 6 = 2a(-1) + b \\ -2 = 2a(5) + b \end{cases}$$

$$\begin{cases} 6 = -2a + b \\ -2 = 10a + b \end{cases}.$$

Subtract the respective sides of these equations to eliminate b.

$$6 - (-2) = -2a - 10a$$

$$8 = -12a$$

$$a = -\frac{2}{3}$$

Substitute this result for a into either of the two equations to determine b.

$$6 = -2\left(-\frac{2}{3}\right) + b$$

$$6 = \frac{4}{3} + b$$

$$b = \frac{14}{3}$$

Plug these values for a and b into equation (1) to determine c.

$$4 = \left(-\frac{2}{3}\right) + \left(\frac{14}{3}\right) + c \quad \rightarrow \quad c = 0$$

Therefore, the equation of the parabola is

$$y = -\frac{2}{3}x^2 + \frac{14}{3}x.$$